Week 3 Worksheet - Partial Fraction

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

Rational function:
Proper rational function:

1. $\int \frac{4}{(3x+1)(x-1)}dx = -\ln|3x+1| + \ln|x-1| + C$

   Work:

$$\frac{4}{(3x+1)(x-1)} = \frac{A}{3x+1} + \frac{B}{x-1}$$
$$4 = A(x-1) + B(3x+1)$$

$$\begin{aligned} \text{Plug in } x &= 1,\ B = 1 \\ \text{Plug in } x &= -\frac{1}{3},\ A = -3 \end{aligned}$$

$$\begin{aligned} \int \frac{4}{(3x+1)(x-1)}dx &= -3\int \frac{1}{3x+1}dx + \int \frac{1}{x-1}dx \\ &= -3\frac{\ln|3x+1|}{3} + \ln|x-1| + C \\ &= -\ln|3x+1| + \ln|x-1| + C \end{aligned}$$

2. $\int \frac{4x^2-4x+6}{x^3-x^2-6x}dx = -\ln|x| + 2\ln|x-3| + 3\ln|x+2| + C$

   Work:

$$x^3 - x^2 - 6x = x(x-3)(x+2)$$

$$\frac{4x^2-4x+6}{x^3-x^2-6x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$
$$4x^2 - 4x + 6 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

$$\begin{aligned} \text{Plug in } x &= 0,\ 6 = -6A \Rightarrow A = -1 \\ \text{Plug in } x &= 3,\ 30 = 15B \Rightarrow B = 2 \\ \text{Plug in } x &= -2,\ 30 = 10C \Rightarrow C = 3 \end{aligned}$$

$$\begin{aligned} \int \frac{4x^2-4x+6}{x^3-x^2-6x}dx &= \int \frac{-1}{x} + \frac{2}{x-3} + \frac{3}{x+2}dx \\ &= -\ln|x| + 2\ln|x-3| + 3\ln|x+2| + C \end{aligned}$$

3. $\int \frac{x^3}{x^2-1}dx = \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$

   Work:

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$\frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$x = A(x+1) + B(x-1)$$
$$A = \frac{1}{2},\ B = \frac{1}{2}$$

$$\begin{aligned}\int \frac{x^3}{x^2-1}dx &= \int x + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}dx \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C\end{aligned}$$

4. $\int \frac{2x^2+3x+7}{x^3+x^2-x-1}dx$

$$\frac{2x^2+3x+7}{x^3+x^2-x-1}$$

Work:

$$x^3+x^2-x-1 = (x+1)\left(x^2-1\right) = (x+1)^2(x-1)$$

$$\frac{2x^2+3x+7}{x^3+x^2-x-1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$
$$2x^2+3x+7 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$
$$x = 1 \Rightarrow 12 = 4C \Rightarrow C = 3$$
$$x = -1 \Rightarrow 6 = -2B \Rightarrow B = -3$$
$$x = 0 \Rightarrow 7 = -A - B + C \Rightarrow A = -B + C - 7 = -1$$

$$\begin{aligned}\int \frac{2x^2+3x+7}{x^3+x^2-x-1}dx &= \int \frac{-1}{x+1} + \frac{-3}{(x+1)^2} + \frac{3}{x-1}dx \\ &= -\ln|x+1| + 3(x+1)^{-1} + 3\ln|x-1| + C\end{aligned}$$

5. $\int \frac{x^2-x-21}{(2x-1)(x^2+4)}dx$

$$\frac{x^2-x-21}{(2x-1)(x^2+4)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}$$
$$x^2-x-21 = A\left(x^2+4\right) + (Bx+C)(2x-1)$$
$$x^2-x-21 = (A+2B)x^2 + (2C-B)x + 4A - C$$

$$\begin{aligned}A+2B &= 1 \\ -B+2C &= -1 \\ 4A-C &= -21\end{aligned}$$

$$\Rightarrow A = -5, B = 3, C = 1$$
$$\int \frac{x^2-x-21}{(2x-1)(x^2+4)}dx = \int \frac{-5}{2x-1} + \frac{3x}{x^2+4} + \frac{1}{x^2+4}dx$$
$$\int \frac{-5}{2x-1}dx = -\frac{5}{2}\ln|2x-1| + C$$

$$\begin{aligned}\int \frac{3x}{x^2+4}dx &= \frac{3}{2}\int \frac{du}{u} \qquad \Big|\ u = x^2+4 \Rightarrow du = 2xdx \\ &= \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln\left|x^2+4\right| + C\end{aligned}$$

$$\begin{aligned}\int \frac{1}{4+x^2}dx &= \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}dx \\ &= \frac{1}{4}\int \frac{1}{1+u^2}2du \qquad \Big|\ u = \tfrac{x}{2} \Rightarrow du = \tfrac{1}{2}dx \\ &= \frac{1}{2}\arctan u + C \\ &= \frac{1}{2}\arctan\frac{x}{2} + C\end{aligned}$$

$$\int \frac{x^2-x-21}{(2x-1)(x^2+4)}dx = -\frac{5}{2}\ln|2x-1| + \frac{3}{2}\ln\left|x^2+4\right| + \frac{1}{2}\arctan\frac{x}{2} + C$$

6. *(from midterm1 2014fall)* $\int \frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)}dx$

(Hint: $x^4+2x^3+3x^2+3x+2 = x(x+1)\left(x^2+x+1\right) + x^2+2x+2$)

$$\frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)} = x + \frac{x^2+2x+2}{(x+1)(x^2+x+1)}$$

$$\frac{x^2+2x+2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$x^2+2x+2 = A\left(x^2+x+1\right) + (Bx+C)(x+1)$$

$$x^2+2x+2 = (A+B)x^2 + (A+B+C)x + A + C$$

$$\begin{aligned}A+B &= 1 \\ A+B+C &= 2 \\ A+C &= 2\end{aligned}$$

$$\Rightarrow A = 1, B = 0, C = 1$$

$$\int \frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)}dx = \int x + \frac{1}{x+1} + \frac{1}{x^2+x+1}dx$$

$$\begin{aligned}\int \frac{1}{x^2+x+1}dx &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx \qquad \left|\ \begin{array}{l}\left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}u^2+\frac{3}{4} \\ x+\frac{1}{2} = \frac{\sqrt{3}}{2}u \Rightarrow dx = \frac{\sqrt{3}}{2}du\end{array}\right. \\ &= \int \frac{1}{\frac{3}{4}u^2+\frac{3}{4}}\frac{\sqrt{3}}{2}du \\ &= \frac{\sqrt{3}}{\frac{3}{2}}\int \frac{1}{u^2+1}du = \frac{2\sqrt{3}}{3}\arctan u + C \\ &= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C\end{aligned}$$

$$\int \frac{x^4+2x^3+3x^2+3x+2}{(x+1)(x^2+x+1)}dx = \frac{x^2}{2} + \ln|x+1| + \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

7. $\int \frac{1}{x^2+6x+10} dx$

Work:

$$x^2 + 6x + 10 = (x+3)^2 + 1$$

$$u = x + 3 \Rightarrow du = dx$$

$$\begin{aligned} \int \frac{1}{u^2+1} du &= \arctan u + C \\ &= \arctan (x+3) + C \end{aligned}$$